Uncoordinated Rate Selection: Approaching the Capacity of Gaussian MAC without Coordination

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***Abstract*—The achievable rate region of a K-user Gaussian multiple access channel (GMAC) is well known. Several schemes have been proposed to achieve some points in this region with minimal encoding and decoding complexities such as successive interference cancelation and successive integer forcing. But these schemes require coordination among the nodes or with the base station for choosing the rates of transmission. In this paper, we propose an uncoordinated scheme in which each node picks the rate *randomly* according to a predetermined probability distribution, independently of the other nodes. We show that in the asymptotic limit as $K \to \infty$, the sumrate achieved by this uncoordinated scheme is only $\Theta(\log \log K)$ away from the sum capacity of GMAC. When an energy constraint is considered instead of a power constraint, we show that our scheme is optimal.**

I. Introduction

THERE has recently been a lot of interest in the design of uncoordinated multiple access (random access) schemes for wireless networks with a large number of users. Potential applications for such schemes are in Machine to Machine communications (M2M), Vehicular Networks and the Internet of Things (IoT). Slotted Aloha is a well known random access scheme that is easy to implement; however, the throughput of slotted Aloha is quite poor when the number of users is large and in particular, the maximum attainable throughput is only $1/e \approx 0.37$ [1].

Recently, there is growing interest in MAC protocols where the collided packets are not discarded, but kept in a buffer and then a successive interference cancelation algorithm is applied to all the received transmissions. Schemes such as *Contention Resolution Diversity Slotted Aloha (CRDSA)* [2], *Irregular repetition Slotted Aloha (IRSA)* [3] [4] were proposed to improve the performance of Slotted Aloha. In these works an analogy was drawn between SIC decoding and Message-passing decoding on an equivalent bipartite graph. Using this idea, it has been shown that when there is no noise in the channel, i.e., only collisions or error-free transmission and when there is no power constraint at the transmitter, a throughput arbitrarily close to one can be obtained by choosing soliton distribution for the choice of repetition rates [5].

In this paper, we consider the important extension of these results to the case when there is additive white Gaussian noise and a transmit power constraint. In this case, the information-theoretic limit on the achievable rate is given by the Gaussian multiple access channel (GMAC) capacity. The corner points of the GMAC region can be achieved with simple successive interference cancelation [6]; however, this requires coordination among the transmitters to choose their rates so that they operate on the corner point of the GMAC rate region. In the absence of such coordination, if the transmitter were to pick their rates so as to lie within the GMAC region, then random coding with joint typicality decoding is optimal. However, the computational complexity of such a scheme would make it infeasible. More recently, spatial coupling has been shown to provide near capacity multiple access in [7], with linear complexity. This still requires some coordination to align the time offsets of the various streams to begin the iterative demodulation process. Spatial coupling has also been used to achieve near-optimal demodulation performance in multiuser systems in [8] and the symmetric rate point of the MAC channel region in [9]. While the complexity of these schemes is linear in the number of users, they still require joint iterative demodulation/decoding which is computationally somewhat complex. In this paper, we introduce a different paradigm where users randomly pick their rates according to a carefully chosen distribution and the decoder decodes the lowest rate user and peels off the user at every stage. With this scheme, we show that even in the absence of coordination, simple single user decoding and successive interference cancelation suffices to obtain close to capacity performance.

II. System Model

We assume there are M users in the network out of which K users have information to transmit to the base station in the current time instant. We call this subset as "active users" and the rest as "idle users". Let us also assume that a time synchronizing mechanism exists between the users and the base station such that time is divided into slots similar to slotted Aloha. Each user waits for the beginning of the time slot to transmit its information. Let $\underline{U}_1, \underline{U}_2, \ldots, \underline{U}_K$ [1] be variable length messages at each active user and $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_K$ be the corresponding n-length codewords generated by using codebooks of rates $\rho_1, \rho_2, \ldots, \rho_K$. The chosen codebooks are assumed to be capacity achieving for a single user point-to-point AWGN channel.

[1] Throughout the paper we use underlined letters to denote vectors.

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We consider two different models for how n and the transmitted power scale with K. In the first model, at each user, the transmit energy per channel use, i.e. power, is constrained so that $\frac{1}{n}\mathrm{E}[\|\underline{X}_i\|^2] \leq P_i, \forall i \in [1, K]$. We call this the finite SNR regime, since in each channel use the SNR seen by each user is finite. Here, the choice of n does not depend on K. In the second model, we assume that n scales linearly with K and the total energy available to user i to transmit its packet is constrained to be P_i' which is finite, and hence per channel use, we impose a power constraint of $P_i = P_i'/n$, i.e., $\mathrm{E}[\|X_{i,j}\|^2] \leq \frac{1}{n}P_i', j = 1, \ldots, n, i = 1, \ldots, K$, where $X_{i,j}$ denotes jth symbol of user i. When $K \to \infty$, this implies that $n \to \infty$ and, hence, the energy per use of the channel is infinitesimally small and we refer to this as the infinitesimal SNR regime. For simplicity, we consider a special case where all the users have identical power constraint i.e. $P_1 = P_2 = \ldots = P_K$.

The noise in the channel is assumed to be Gaussian with zero mean and variance σ^2. The achievable rate region for a GMAC is given by the following set of K inequalities[6], where each set for $j \in [1 : K] := \{1, 2, \ldots, K\}$ is given by :

$$\sum_{i \in \mathcal{S}} \rho_i \leq \log_2 \left(1 + \frac{\sum_{i=1}^{j} P_i}{\sigma^2}\right) \qquad \forall \mathcal{S} \subseteq [1 : K], |\mathcal{S}| = j, \tag{1}$$

where ρ_i is the rate picked by user i. The rate region is a subspace of $\mathbb{R}^K$. A point in $\mathbb{R}^K$ represents a K-tuple of rates chosen by all users. Some points in this rate region are "easily achieved" compared to others. For example the following set of corner points are achievable by a simple iterative decoder, which has the complexity of a single user decoder.

$$\rho_i \leq \log_2 \left(1 + \frac{P_i}{\sigma^2 + \sum_{j=1}^{i-1} P_j}\right) \qquad i = 1, 2, ..., K \tag{2}$$

For rates chosen according to (2), this $K \times 1$ MIMO channel is transformed into K-level SISO channels. The decoding starts with the user whose rate is the lowest, while treating all other codewords as Gaussian noise. After successful decoding, the codeword is subtracted from the received message and the process continues to the higher levels [10]. This kind of decoding process is often referred to as *Onion peeling* or *Successive Interference Cancelation* (SIC).

III. Uncoordinated Rate Selection

In our proposed scheme each user maintains a set of L codebooks, each with rates $\{r_1, r_2, \ldots, r_L\}$. The target is to operate near the corner points of the rate region (2), since this rate point can be achieved with the complexity of single-user decoding. The users must choose the codebooks and thus the rates without any coordination. In our scheme, each user chooses the rate from a predetermined rate distribution, which depends only on the number of active users in the network, statistical properties of the Gaussian noise in the channel and the probability of decoder failure that can be tolerated. The message is then encoded using the codebook of the chosen rate. The SIC decoder iteratively decodes each message transmitted in that time slot. The following definitions are used throughout the paper.

Definition 1. *(Rate Distribution) A discrete function that gives the probability that a particular rate is chosen by the user.*

Definition 2. *(Capacity Achieving) A scheme is said to be capacity achieving if the sum of rates of all the users transmitting in one channel use is equal to the sum capacity of GMAC, S_{cap}, which is equal to $\log_2\left(1 + \frac{KP}{\sigma^2}\right)$ bits/ch. use.*

Definition 3. *(Gap to Capacity) We define two metrics to measure the gap between sumrate achieved by the proposed scheme $S := \sum_{i=1}^{K} \rho_i$ and the sum capacity, S_{cap}: (i) Absolute gap, $g_a := S_{cap} - S$ and (ii) Fractional gap, $g_f := \frac{S}{S_{cap}}$.*

A. Rate Quantization

Rate quantization is an operation in which an L-length rate distribution is chosen such that it is within the achievable rate region of GMAC. Ideally L must be equal to K to operate at any corner point in the rate region. But we choose $L < K$ since it provides additional degree of freedom in the asymptotic analysis which we will see in section IV. In this work, we propose two ways to perform rate quantization namely, Uniform quantization and Optimal quantization. While the former is more suitable for the asymptotic analysis, the latter achieves higher sum rates for finite K.

1) Uniform Quantization: We assume that the number of users picking each of the quantized rates is uniform. Let n_j be the number of users picking rate r_j. Uniform quantization implies, $n_1 = n_2 = \ldots = n_L = \frac{K}{L}$. For the given $\{n_j\}$, we can calculate the rates $\{r_j\}$ such that they satisfy the constraints imposed by the SIC decoder (2):

$$r_1 = \log_2 \left(1 + \frac{P}{(K-1)P + \sigma^2}\right) \tag{3}$$

$$r_j = \log_2 \left(1 + \frac{P}{\left(K - (j-1)\frac{K}{L} - 1\right)P + \sigma^2}\right) j = 2, ..., L$$

2) Optimal Quantization: The objective is to maximize the *sumrate*, subject to the constraint imposed by the SIC decoder (2). Each user with rate r_j has interference from all the other users with rates $r_k > r_j$, which can be treated as noise by the SIC decoder. Hence we can formulate this quantization operation as the following optimization problem:

$$\begin{aligned} \underset{n_j, r_j \forall j \in [1:L]}{\text{maximize}} \quad & \sum_{j=1}^{L} n_j r_j \\ \text{subject to} \quad & \sum_{j=1}^{L} n_j = K \\ & r_1 \leq \log_2 \left(1 + \frac{P}{(K-1)P + \sigma^2}\right) \\ & r_j \leq \log_2 \left(1 + \frac{P}{\left(K - \sum_{k=1}^{j-1} n_k - 1\right)P + \sigma^2}\right) \\ & r_j \geq 0 \qquad j = 1, 2, ..., L \end{aligned}$$

$$n_j \geq 0 \tag{4}$$

The above optimization problem is non-convex in n_j and r_j. However, we can find a suboptimal solution by finding a local maximum. We use the approximation $\ln(1+x) \approx x$ for $x \approx 0$, which is valid in the low SNR regime, to derive a closed form solution using the method of Lagrange multipliers. This solution gives only a local maximum.

$$\begin{aligned}
n_j^* &= \frac{1}{(A\lambda)^{L-j-1}} \left[\frac{1}{A\lambda^2} - \frac{1}{\lambda}\right] \qquad j = 1, 2, ...L-1 \\
n_L^* &= K - \sum_{k=1}^{L-1} n_k^* \\
r_1^* &= \frac{1}{\ln 2}\left(\frac{P}{(K-1)P + \sigma^2}\right) \\
r_j^* &= \frac{1}{\ln 2}\left(\frac{P}{\left(K - \sum_{k=1}^{j-1} n_k^* - 1\right)P + \sigma^2}\right) \quad j = 2, ..., L \\
&\text{where} \quad A = \frac{\sigma^2}{P}, \lambda = \left(\frac{1}{A^{L-1}(K-1+A)}\right)^{1/L}
\end{aligned} \tag{5}$$

Definition 4. *(SIC failure) Let* e_j *denote the actual number of users picking rate* r_j^* *and let* $\underline{E} = \{e_j\}$ *denote the empirical distribution. Since users pick rates independently, for finite* K, $\underline{E}$ *will deviate from the desired distribution* $\{n_j^*\}$ *and, hence, cause the SIC decoder to fail with some probability. The decoder fails during stage* m *(*$m < L$*), when*

$$e_1 + e_2 + \ldots + e_m < \frac{\sum_{k=1}^{m} n_k^*}{K}. \tag{6}$$

The above equation means that there are not enough number of lower rate users to decode to proceed to stage $m+1$.

B. Rate Biasing

To increase the number of users picking lower rates, we need to bias the rate distribution obtained from the quantization operation by increasing the weights on the lower rate users. We refer to this operation as Rate biasing. Let $\underline{Q}$ be the resultant biased distribution. The objective is to maximize the expected sum rate, $\sum_{i=1}^{K} q_i r_i^*$, subject to an upper bound on the probability of decoding failure P_{fail}. We can formulate this as an optimization problem:

$$\begin{aligned}
\underset{q_j \forall j \in [1,L]}{\text{maximize}} \quad & \sum_{j=1}^{L} q_j r_j^* \\
\text{subject to} \quad & \sum_{j=1}^{L} q_j = 1 \\
& P_{\text{fail}} \leq \hat{\epsilon}
\end{aligned} \tag{7}$$

The probability of decoding failure is analyzed using the Sanov's theorem, which gives a bound on the probability of observing an atypical sequence of samples. It states that if $X_1, X_2, ...X_n$ are i.i.d. random variables $\sim \underline{Q}$ and E is any set of empirical distributions, then

$$Q^n(E) \leq (n+1)^{|\mathcal{X}|} 2^{-nD(\underline{E}_i^*||\underline{Q})} \tag{8}$$

where $Q^n(E)$ is the probability that an empirical distribution in the set E resulted by taking n samples following the probability distribution $\underline{Q}$ and $\underline{E}^*$ is the empirical distribution in E that is closest to $\underline{Q}$ in terms of Kullback-Liebler distance.

We define E as a set of all the distributions in the L-dimensional space that cause the iterative decoding to fail. Thus $Q^K(E)$ is the probability of decoding failure. When the iterative decoding fails in stage m whose condition is given by (6), we refer to it as failure mode m. To calculate P_{fail} for mode m using Sanov's theorem, we need to calculate $\underline{E}_m^*$, which is the empirical distribution closest in KL-distance to $\underline{Q}$ in the set of E that has a failure mode m. We formulate it as an optimization problem:

$$\begin{aligned}
\underset{e_j \forall j \in [1,L]}{\text{minimize}} \quad & D(\underline{E}||\underline{Q}) \\
\text{subject to} \quad & \sum_{j=1}^{L} e_j = 1 \\
& e_1 + \ldots + e_m \leq \sum_{k=1}^{m} \frac{n_k^*}{K}
\end{aligned} \tag{9}$$

The value of the objective function of the above optimization problem gives $D(\underline{E}_m^*||\underline{Q})$. The closed form solution can be obtained using the method of Lagrange Multipliers.

$$\underline{E}_m^*(j) = \begin{cases} \dfrac{q_j \sum_{k=1}^{m} n_k^*}{\sum_{k=1}^{m} q_k} & \text{for } j = 1, 2, \ldots, m \\[2ex] \dfrac{q_j \sum_{k=m+1}^{L} n_k^*}{\sum_{k=m+1}^{L} q_k} & \text{for } j = m+1, \ldots, L \end{cases} \tag{10}$$

The constraint $P_{\text{fail}} \leq \hat{\epsilon}$ can be shown equivalent to $D(\underline{E}_m^*||\underline{Q}) \geq \epsilon \ \forall m \in [1:L]$, for some ϵ dependent on $\hat{\epsilon}$ whose relation is derived in Theorem 2. Intuitively, it means that if each set of empirical distributions that result in decoding failure have a minimum distance of ϵ from $\underline{Q}$, we can certify that the iterative decoding succeeds with a probability of at least $1 - \hat{\epsilon}$. Since we know $\underline{E}_m^*$ in terms of $\underline{Q}$ as shown above, the condition $D(\underline{E}_m^*||\underline{Q}) > \epsilon$ can be simplified to:

$$\begin{aligned}
& sq(m)^{sn(m)} \left(1 - sq(m)\right)^{1-sn(m)} \leq \\
& \qquad e^{-\epsilon}\left[sn(m)^{sn(m)} \left(1 - sn(m)\right)^{1-sn(m)}\right]
\end{aligned} \tag{11}$$

Where $sn(m) = \sum_{k=1}^{m} n_k^*$ and $sq(m) = \sum_{k=1}^{m} q_k$ are the cumulative sums of $\underline{N}$ and $\underline{Q}$. It can easily be shown that the inequality in (11) has only two feasible regions:

$$sq(m) \leq a_{m1} \text{ or } sq(m) \geq a_{m2} \tag{12}$$

where a_{m1}, a_{m2} are the two roots of the above equation. This discontinuous choice of Q-region is difficult to handle while solving the optimization problem in (9). But it turns out that the condition $sq(m) \leq a_{m1}$ results in a $\underline{Q}$ that is already in E, the set of distributions that cause decoding failure. Thus the constraint $D(\underline{E}_m^*||\underline{Q}) > \epsilon$ is equivalent to $sq(m) \geq a_{m2}$, which results in a linear optimization problem.

$$\begin{aligned}
\underset{q_j \forall j \in [1,L]}{\text{maximize}} \quad & \sum_{j=1}^{L} q_j r_j \\
\text{subject to} \quad & \sum_{j=1}^{L} q_j = 1 \\
& \sum_{j=1}^{m} q_j \geq a_{m2} \qquad m = 1, 2, \ldots, L-1
\end{aligned} \tag{13}$$

IV. Asymptotic Analysis

We use $S_{\text{uq}}, S_{\text{oq}}$ to represent maximum sumrates obtained through uniform quantization and optimal quantization respectively, $S_{\text{uq,sim}}, S_{\text{oq,sim}}$ to represent sumrates obtained at the decoder by choosing their corresponding biased distributions. We begin our analysis by noting that there is a loss of *sumrate* in two stages of our scheme. In the first stage, Rate quantization, since $L \leq K$, $S_{\text{uq}} \leq S_{\text{cap}}$. In the second stage due to the biasing operation, for any $\epsilon \geq 0$, we have $S_{\text{sim}} \leq S_{\text{uq}}$. We characterize the dependence of first loss on the number of quantization levels, L, as $K \to \infty$ in the following theorem:

Theorem 1. *For a fixed L, as $K \to \infty$, the upper bound on sumrate achieved through uniform quantization is $\Theta(\log_2 L)$*

Proof. Let $A = \text{SNR}^{-1} = \frac{\sigma^2}{P}$. Then,

$$\begin{aligned}
S_{\text{uq}} &= \frac{K}{L} \log_2 \left(1 + \frac{1}{K-1+A}\right) \\
&+ \sum_{i=2}^{L} \frac{K}{L} \log_2 \left(1 + \frac{1}{K-(i-1)\frac{K}{L} - 1 + A}\right)
\end{aligned} \tag{14}$$

$$\lim_{K\to\infty} \frac{K}{L} \log_2 \left(1 + \frac{1}{K-1+A}\right) = \frac{1}{L \ln 2}$$

$$\begin{aligned}
\lim_{K\to\infty} \frac{K}{L} \log_2 &\left(1 + \frac{1}{K-(i-1)\frac{K}{L} - 1 + A}\right) \\
&= \frac{1}{L \ln 2} \left(\frac{1}{1 - \left(\frac{i-1}{L}\right)}\right)
\end{aligned}$$

Using the following upper bound for the summation of harmonic series an upper bound is derived for S_{uq}. Here γ is the EulerMascheroni constant and $\epsilon_k \approx \frac{1}{2k}$

$$\sum_{n=1}^{k} \frac{1}{n} = \ln k + \gamma + \epsilon_k < \ln k + 1 \tag{15}$$

$$\begin{aligned}
\lim_{K\to\infty} S_{\text{uq}} &= \frac{1}{L \ln 2} + \sum_{i=2}^{L} \frac{1}{L \ln 2} \left(\frac{1}{1 - \left(\frac{j-1}{L}\right)}\right) \\
&< \frac{1}{L \ln 2} + \frac{1}{L \ln 2} (L \ln L) \\
&= \Theta(\log_2 L)
\end{aligned}$$

□

Theorem 2. *For a fixed L, as $K \to \infty$, the probability of decoder failure goes to zero.*

Proof. Let $P_{f,m}$ be the probability that the iterative decoder fails in mode m. From Sanov's theorem, we have,

$$P_{f,m} \leq (K+1)^L \, 2^{-KD\left(\underline{E}_m^* || \underline{Q}\right)} \tag{16}$$

From (7), which bounds P_{fail}, we have:

$$\begin{aligned}
\sum_{i=1}^{L-1} P_{f,i} &\leq \widehat{\epsilon} \\
\sum_{i=1}^{L-1} \frac{(K+1)^L}{2^{K\epsilon}} &\leq \widehat{\epsilon} \\
\epsilon &\geq \frac{1}{K} \log_2 \frac{(L-1)(K+1)^L}{\widehat{\epsilon}} \\
\epsilon &\geq \frac{1}{K} \left[\log_2(L-1) - \log_2 \widehat{\epsilon} + L \log_2(K+1)\right] \quad (17) \\
\lim_{K\to\infty} \epsilon &\geq 0
\end{aligned}$$

Hence for a sufficiently large K, there exists an arbitrarily small δ such that $\epsilon \to \delta$ can guarantee arbitrarily small decoder failure probability, γ, where $\gamma = O(K^{-c}), \forall c \geq 0$. Thus the probability of decoder failure decays as the polynomial power of K. This implies that for a fixed L, the rate biasing operation can be bypassed in the asymptotic limit as $K \to \infty$. □

From theorems 1 and 2, it is evident that L should grow as a function of K such that the total loss in sum rate is minimal. We choose $L = \frac{K}{((c+1)\cdot \log_2 K)^{1+\delta}}$ for an arbitrarily small $\delta > 0$. The choice of c determines how fast the probability of decoder failure decays with the number of users, $\widehat{\epsilon} = O(K^{-c})$.

Theorem 3. *In the finite SNR regime the lower and upper bounds for S_{uq} are:*

$$\frac{K}{L} \cdot \frac{\log_2 \mathrm{e}}{K - \frac{1}{2} + A} + \log_2 \left[\frac{\frac{L}{K}(K - \frac{1}{2} + A) - 1}{\frac{L}{K}(K - \frac{1}{2} + A) - L + 1}\right] - \log_2 \mathrm{e} \leq$$

$$S_{uq} \leq \frac{K}{L} \cdot \frac{\log_2 \mathrm{e}}{K - 1 + A} + \log_2 \left[\frac{L}{K} \cdot (K - 1 + A) - 1\right] + \log_2 \mathrm{e}$$

Proof. From (14) we have an analytical form for S_{uq}. Using the approximation, $\frac{2x}{2+x} \leq \ln(1+x) \leq x$, upper and lower bounds are calculated separately for each of the two parts.

$$\frac{K}{L} \cdot \frac{\log_2 \mathrm{e}}{K - \frac{1}{2} + A} \leq \frac{K}{L} \log_2 \left[1 + \frac{1}{K-1+A}\right] \leq \frac{K}{L} \cdot \frac{\log_2 \mathrm{e}}{K - 1 + A}$$

Using the harmonic series approximation in (15), we have the bounds for the second part.

$$\begin{aligned}
&\log_2 \left[\frac{\frac{L}{K}(K - \frac{1}{2} + A) - 1}{\frac{L}{K}(K - \frac{1}{2} + A) - L + 1}\right] - \log_2 \mathrm{e} \\
&\leq \sum_{i=2}^{L} \frac{K}{L} \log_2 \left[1 + \frac{1}{K - (i-1)\frac{K}{L} - 1 + A}\right] \\
&\leq \log_2 \left[\frac{L}{K}(K - 1 + A) - 1\right] + \log_2 \mathrm{e}
\end{aligned}$$

□

Theorem 4. *In the finite SNR regime, for the choice of $L = \frac{K}{(\log_2 K)^{1+\delta}}$, the absolute gap and the fractional gap of S_{uq} from S_{cap} in the asymptotic limit as $K \to \infty$ are:*

$$\begin{aligned}
\lim_{K\to\infty} g_a &= \Theta(\log \log K) \\
\lim_{K\to\infty} g_f &= 1
\end{aligned}$$

Proof. We observe the upper and lower bounds of g_a derived from the previous theorem in the asymptotic limit as $K \to \infty$.

$$\lim_{K\to\infty} g_a = \log_2(1+\frac{K}{A}) - S_{\text{uq}}$$

$$\begin{aligned}\lim_{K\to\infty} g_a &\geq \lim_{K\to\infty}\Bigg[\log_2(1+\frac{K}{A}) + \log_2 \mathrm{e} \\ &\quad - \log_2\left[\frac{\frac{L}{K}(K-\frac{1}{2}+A)-1}{\frac{L}{K}(K-\frac{1}{2}+A)-L+1}\right]\Bigg] \\ &= \log_2\left[\lim_{K\to\infty}\frac{(1+\frac{K}{A})(\frac{L}{K}(-\frac{1}{2}+A)+1)}{\frac{L}{K}(K-\frac{1}{2}+A)-1}\right] + \log_2 \mathrm{e} \\ &= \log_2\left[(1+\delta)\log_2 K\right] + \log_2 \mathrm{e} - \log_2 A\end{aligned}$$

Similarly we can show that,

$$\lim_{K\to\infty} g_a \leq \log_2\left[(1+\delta)\log_2 K\right] - \log_2 \mathrm{e} - \log_2 A$$

In the asymptotic limit as $K \to \infty$, the bounds converge and hence $g_a = \Theta(\log\log K)$.

$$\lim_{K\to\infty} g_f = \lim_{K\to\infty}\left[1 - \frac{g_a}{S_{\text{cap}}}\right] = 1 \qquad \square$$

Theorem 5. *In the infinitesimal SNR regime, the lower and upper bounds for S_{uq} are:*

$$\log_2\left[\frac{\frac{L}{K}(K-\frac{1}{2}+A')-1}{\frac{L}{K}(K-\frac{1}{2}+A')-L}\right] - \log_2 \mathrm{e} \leq$$

$$S_{uq} \leq \log_2\left[\frac{\frac{L}{K}(K-1+A')-1}{\frac{L}{K}(K-1+A')-L}\right] + \log_2 \mathrm{e}$$

Proof. Here A' is the inverse SNR, which is directly proportional to n and, hence, to K. The proof is similar to theorem 3 with a slight variation in the harmonic series approximation. $\square$

Theorem 6. *In the infinitesimal SNR regime, for the choice of $L = \frac{K}{(\log_2 K)^{1+\delta}}$, the absolute gap of S_{uq} from S_{cap} in the asymptotic limit as $K \to \infty$ is bounded by:*

$$0 \leq \lim_{K\to\infty} g_a \leq \log_2 \mathrm{e}$$

Proof. As $K \to \infty$, $n \to infty$ and SNR $\to 0$ such that $\frac{K}{A'} = d$, where d is constant.

$$\begin{aligned}\lim_{\substack{K\to\infty\\ \frac{K}{A'}\to d}} g_a &\leq \log_2(1+d) + \log_2 \mathrm{e} \\ &\quad - \lim_{\substack{K\to\infty\\ \frac{K}{A'}\to d}} \log_2\left[\frac{\frac{L}{K}(K-\frac{1}{2}+A')-1}{\frac{L}{K}(K-\frac{1}{2}+A')-L}\right] \\ &= \log_2(1+d) + \log_2 \mathrm{e} + \log_2\left[\lim_{K\to\infty}\frac{\frac{K}{d}-\frac{1}{2}}{K-\frac{1}{2}+\frac{K}{d}-(\log_2 K)^{1+\delta}}\right] \\ &= \log_2 \mathrm{e}\end{aligned}$$

Similarly we can obtain a lower bound of $(-\log_2 \mathrm{e})$ but zero is a trivial tighter lower bound. $\square$

Remark. *Despite the looseness of the above bounds, we observe in simulations that the absolute gap does go to zero.*

$$\begin{aligned}\lim_{\substack{K\to\infty\\ \frac{K}{A'}\to d}} g_f &= 1 - \frac{g_a}{S_{cap}} \\ &\geq 1 - \frac{\log_2 \mathrm{e}}{\log_2(1+d)}\end{aligned}$$

When d is large, equivalently, when P/σ^2 is large, g_f can be arbitrarily close to 1.

V. Results

We aim to substantiate our theoretical analysis in section IV by setting up the following simulation scenarios.

A. Role of ϵ: Finite SNR regime

In Fig. 1, the sum rate obtained from simulations (in red) is plotted as a function of the biasing parameter ϵ for a fixed K and L. It is evident from the plot that large value of ϵ implies excessive biasing and hence lower sumrate, whereas, small value of epsilon implies improper biasing and hence higher probability of decoder failure. Thus there is an optimal choice of ϵ that gives the maximum sumrate.

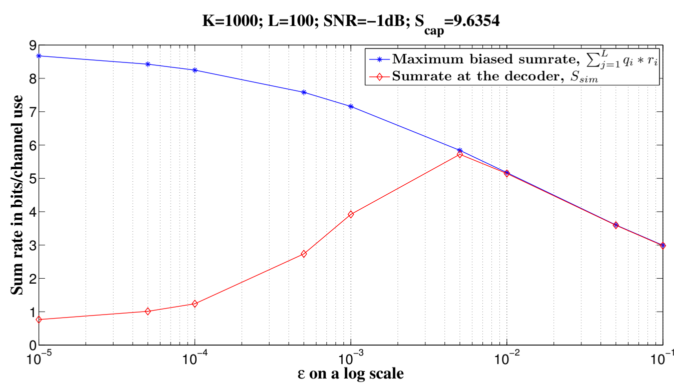

Fig. 1. Role of ϵ in the Rate biasing operation

B. Uniform Quantization vs Optimal Quantization

In Table I, the sum rate obtained with and without biasing is given for both Uniform quantization and Optimal quantization. For finite K, it can be seen that Optimal quantization results in higher sumrates compared to Uniform quantization. However, this difference becomes insignificant for large values of K.

TABLE I
K=1000, GMAC CAPACITY=9.6354

L	S_{uq}	S_{oq}	$S_{\text{uq,sim}}$	$S_{\text{oq,sim}}$
10	4.2088	7.3657	3.7729	4.6621
100	7.3153	10.0988	5.5257	6.0682
1000	9.6354	10.4469	6.0508	6.2844

From Table I, note that S_{oq}, which is the sum rate obtained from optimal quantization, is greater than the capacity. This happens because the decision variables in the optimization problem $\{n_i\}$ are constrained to be integers, but solving a mixed integer problem is hard. Relaxing the integer constraint gives a suboptimal solution. It is of significant interest to find a global optimum for the quantization problem with integer constraints on $\{n_i\}$.

C. *Asymptotic behavior: Fixed L, $K \to \infty$*

To validate Theorems 1 & 2, we fix L and increase K to observe $S_{\text{uq}}, S_{\text{oq}}$and $S_{\text{oq,sim}}$. Since K is increasing exponentially, Theorem 1 predicts that the gap between S_{cap} and S_{uq} increases in the order of $\Theta(\log_2 K)$ and hence linearly in this case. It can be seen from Table II, columns 2 and 3 that this is true. Theorem-2 also predicts that ϵ goes to zero as $K \to \infty$, which can be seen as true from column 5.

TABLE II
SNR=-40 dB, L=100, Asymptotic behavior of K

K	GMAC capacity	S_{oq}	$S_{\text{oq,sim}}$	ϵ
10^2	0.0144	0.0144	0.0143	9×10^{-2}
10^3	0.1375	0.1374	0.1368	1×10^{-2}
10^4	1	0.9966	0.9857	1×10^{-3}
10^5	3.4594	3.3710	3.3760	5×10^{-5}
10^6	6.6582	6.4328	6.4355	11×10^{-6}
5×10^6	8.9687	8.6956	8.6175	1.5×10^{-6}

D. *g_a, g_f in the finite SNR regime*

For the choice of $L = \frac{K}{(\log_2 K)^2}$, we studied the absolute and fractional gaps to validate Theorems 3 and 4 and these results are reported in Table III. From column 5, we observe that $\epsilon \to 0$ hence $S_{\text{sim}} \to S_{\text{uq}}$. We also observe that the fractional gap progresses as $\{0.31, 0.39, 0.47, 0.54, 0.56, 0.58\}$. This data is far from conclusive that $g_f \to 1$ as $K \to \infty$. But it quickly becomes infeasible to simulate for such large values of K. The absolute gap progresses as $\{4.359, 5.87, 6.87, 7.52, 7.72, 7.98\}$. This validates Theorem 4, which states that $\lim_{K\to\infty} g_a = \Theta(\log\log K)$.

TABLE III
Asymptotic behavior for increasing L. SNR=-1 dB, $L = \frac{K}{(\log_2 K)^2}$

K	L	S_{cap}	S_{uq}	S_{sim}	ϵ
10^2	2	6.3297	2.1371	1.9711	3×10^{-2}
10^3	10	9.6354	4.2088	3.7644	5×10^{-3}
10^4	57	12.9557	6.6681	6.0898	1×10^{-3}
10^5	362	16.2780	9.3281	8.7594	11×10^{-5}
3×10^5	906	17.8624	10.6515	10.1390	3×10^{-5}
6×10^5	1629	18.8624	11.4981	10.8846	1.2×10^{-5}

E. *g_a, g_f in the infinitesimal SNR regime*

In Table IV, we report results for the infinitesimal regime. From column 5, we observe that $\epsilon \to 0$ hence $S_{\text{sim}} \to S_{\text{uq}}$. We also observe that the fractional gap progresses as $\{0.8544, 0.9630, 0.9921, 0.9983\}$, which supports our claim that URS is optimal in the infinitesimal SNR regime. Although Theorem 6 does not imply optimality due to the looseness of the bounds, this simulation validates our claim.

TABLE IV
Asymptotic behavior for increasing L. SNR (P'/σ^2)=-1dB, $L = \frac{K}{(\log_2 K)^2}$

K	L	S_{cap}	S_{uq}	S_{sim}	ϵ
10^2	2	0.8434	0.7313	0.7206	1×10^{-2}
10^3	10	0.8434	0.8123	0.8122	1×10^{-3}
10^4	57	0.8434	0.8390	0.8367	1×10^{-4}
10^5	362	0.8434	0.8427	0.8420	1×10^{-5}

VI. Conclusion

Our proposed scheme, URS, provides the mathematical framework required to obtain the rate PMF such that each node in the network can choose rates independently and still achieve sumrates close to the sum capacity of GMAC. In this work it has been shown analytically that the gap to capacity is $\Theta(\log\log K)$ in the finite SNR regime and is capacity approaching in the infinitesimal SNR regime. Although it is not discussed in detail, it is possible to extend iterative collision resolution (ICR) to the AWGN channel by using a capacity achieving code for single user AWGN channel, the absolute gap to sum capacity in the finite SNR regime is $\Theta(\log\log K)$. But ICR performs poorly in the infinitesimal SNR regime, i.e., $g_f \to 0$ and our proposed scheme has an advantage over ICR in this regime.

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